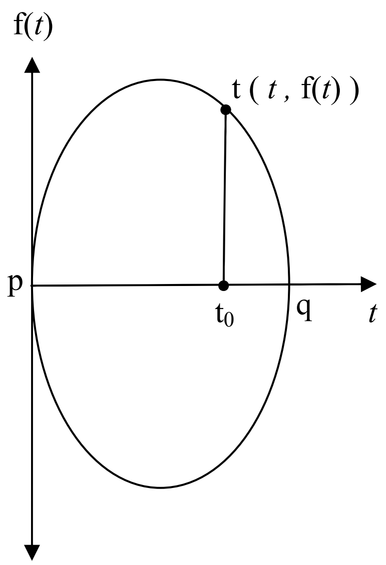
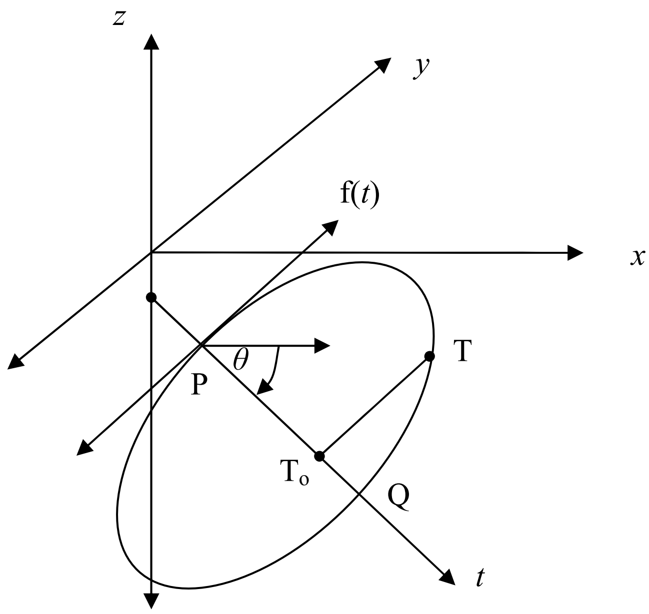
The following describes a possible model for steel pans (or steelpans or steel drums). It assumes that all the notes have two axes of symmetry, the note boundary (groove) lies on a plane and the shape of the largest note (on the rim and in the centre) is a superellipse or Lamé Oval. (See http://mathworld.wolfram.com/Superellipse.html). The pan profile between notes is determined by the large note and remains the same around the pan. All smaller note locations and boundaries are generated from the profile, so that two axes of symmetry are maintained for all notes.

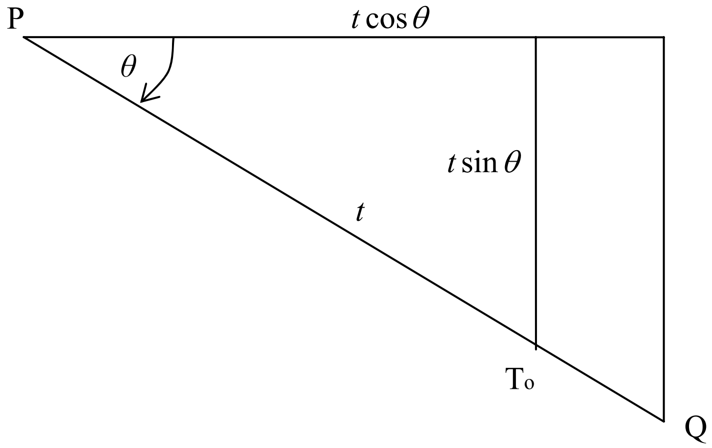




Given a note shape with points (*t*, f(*t*)) translate p to P and rotate through θ. $\quad \mathrm{pt}_0 = \mathrm{PT}_0$

$\mathrm{P} = (x_\mathrm{P}, 0, z_\mathrm{P})$, $\quad \mathrm{Q} = (x_\mathrm{Q}, 0, z_\mathrm{Q})$, $\quad \mathrm{T} = (x_\mathrm{T}, y_\mathrm{T}, z_\mathrm{T})$, $\quad \mathrm{T}_0 = (x_\mathrm{T}, 0, z_\mathrm{T})$

T is any point on the edge of the note.



$$\mathrm{T}_0 = \left(x_\mathrm{P} + t\cos\theta, 0, z_\mathrm{p} - t\sin\theta\right)$$

$$\mathrm{T} = \left(x_\mathrm{p} + t\cos\theta, \mathrm{f}(t), z_\mathrm{p} - t\sin\theta\right)$$

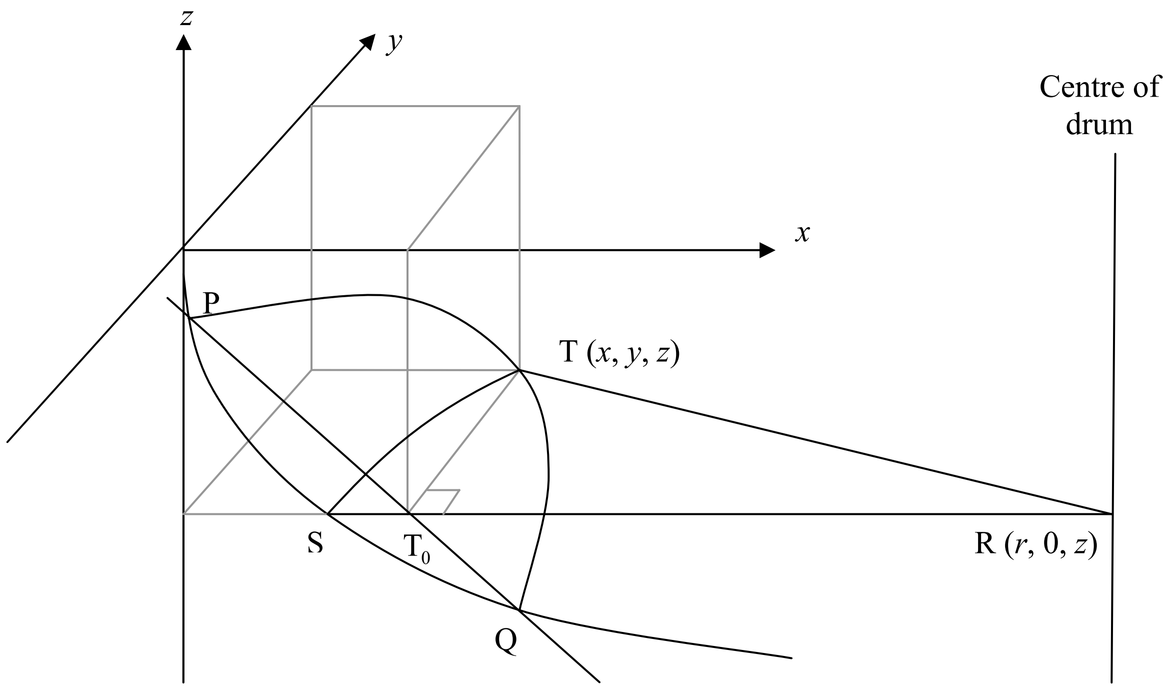
$$z_\mathrm{T} = z_\mathrm{P} - t\sin\theta$$

$$t = \frac{z_\mathrm{P} - z_\mathrm{T}}{\sin\theta}$$

$$\therefore x_\mathrm{T} = x_\mathrm{p} + \frac{z_\mathrm{P} - z_\mathrm{T}}{\sin\theta} \cdot \cos\theta$$

$$= x_\mathrm{p} + \frac{z_\mathrm{P} - z_\mathrm{T}}{\tan\theta}$$

$$y_\mathrm{T} = \mathrm{f}(t) = \mathrm{f}\left(\frac{z_\mathrm{P} - z_\mathrm{T}}{\sin\theta}\right)$$



If $PQ = 2a$, then $Q = \left(x_p + 2a\cos\theta, 0, z_p - 2a\sin\theta\right)$

and midpoint of PQ: $\dfrac{PQ}{2} = M = \left(x_p + a\cos\theta, 0, z_p - a\sin\theta\right)$

$RS = RT$

$$= \sqrt{(r-x)^2 + y^2}$$
$$= \sqrt{(r - x_P - t\cos\theta)^2 + [f(t)]^2}$$
$$= \sqrt{\left(r - x_P - \frac{z_P - z}{\tan\theta}\right)^2 + \left[f\left(\frac{z_P - z}{\sin\theta}\right)\right]^2}$$
$$S = \left(r - \sqrt{(r - x_P - t\cos\theta)^2 + [f(t)]^2}\ ,\ 0\ ,\ z_P - t\sin\theta \right)$$
$$S = \left(r - \sqrt{\left(r - x_P - \frac{z_P - z}{\tan\theta}\right)^2 + \left[f\left(\frac{z_P - z}{\sin\theta}\right)\right]^2}\ ,\ 0\ ,\ z \right)$$

Summary: Given depth z, and note shape function f(t), and angle θ (>0) equal to the note plane angle measured down from the horizontal plane (i.e. the x axis), any edge point T on the note is $T = \left(x_p + t\cos\theta, f(t), z_p - t\sin\theta\right) = \left(x_p + \dfrac{z_P - z}{\tan\theta}\ ,\ f\left(\dfrac{z_P - z}{\sin\theta}\right)\ ,\ z \right)$.

Given z, the profile between notes is function F(z).

$$x = F(z) = r - \sqrt{(r - x_P - t\cos\theta)^2 + [f(t)]^2} = r - \sqrt{\left(r - x_P - \frac{z_P - z}{\tan\theta}\right)^2 + \left[f\left(\frac{z_P - z}{\sin\theta}\right)\right]^2} \qquad \text{equ 1}$$

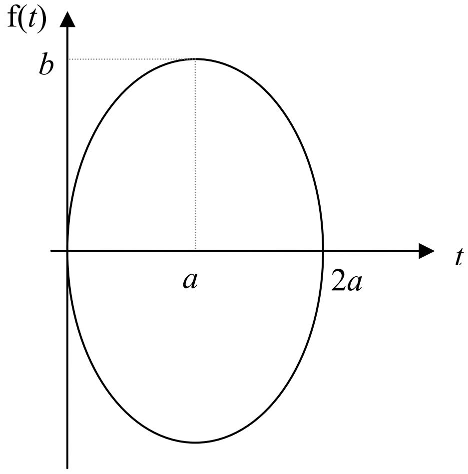
The derivative of the profile equation is

$$\frac{dx}{dz}=\frac{\frac{2}{\tan\theta}\left(r-x_{\mathrm{P}}-\frac{z_{\mathrm{P}}-z}{\tan\theta}\right)+2\left[\mathrm{f}\left(\frac{z_{\mathrm{P}}-z}{\sin\theta}\right)\right]\cdot\frac{d}{dz}\left[\mathrm{f}\left(\frac{z_{\mathrm{P}}-z}{\sin\theta}\right)\right]}{-2\sqrt{\left(r-x_{\mathrm{P}}-\frac{z_{\mathrm{P}}-z}{\tan\theta}\right)^2+\left[\mathrm{f}\left(\frac{z_{\mathrm{P}}-z}{\sin\theta}\right)\right]^2}}$$

$$\frac{dx}{dz}=\frac{\left(r-x_{\mathrm{P}}-\frac{z_{\mathrm{P}}-z}{\tan\theta}\right)+\tan\theta\left[\mathrm{f}\left(\frac{z_{\mathrm{P}}-z}{\sin\theta}\right)\right]\cdot\frac{d}{dz}\left[\mathrm{f}\left(\frac{z_{\mathrm{P}}-z}{\sin\theta}\right)\right]}{(x-r)\tan\theta}\qquad \text{equ 2}$$

Maximum depth is at $\frac{dz}{dx}=0$, that is, when $x=r$

Note Equations



Tangential notes for double seconds, guitars $b>a$
Radial notes for leads $a>b$:

$$\left|\frac{a-t}{a}\right|^m + \left|\frac{f(t)}{b}\right|^n = 1, \quad b \geq a, \quad m,n \geq 2$$

$$\left|\frac{f(t)}{b}\right|^n = \left(1 - \left|\frac{a-t}{a}\right|^m\right)$$

$$f(t) = \pm b\left(1 - \left|\frac{a-t}{a}\right|^m\right)^{\frac{1}{n}}, 0 < t \leq 2a \qquad \text{equ3}$$

$$\text{equ3a} \quad f\left(\frac{z_P - z}{\sin\theta}\right) = \pm b\left(1 - \left|1 - \frac{z_P - z}{a\sin\theta}\right|^m\right)^{\frac{1}{n}} \quad z_P \geq z \geq z_P - 2a\sin\theta$$

Since the note lies in the dish shape of the pan, z extends beyond this range. The profile equation F(z) includes the term $\left[f\left(\frac{z_P - z}{\sin\theta}\right)\right]^2$ which indicates that the best value for n is 2.

If $n = 2$, $\left[f\left(\frac{z_P - z}{\sin\theta}\right)\right]^2 = b^2\left(1 - \left|1 - \frac{z_P - z}{a\sin\theta}\right|^m\right) \leq 0$ when $z \leq z_P - 2a\sin\theta$ or $z \geq z_P$.

≥ 0 when $z_P \geq z \geq z_P - 2a\sin\theta$

For other values of n, absolute values take care of non-integer exponents of negative numbers, i.e.

$$\left[f\left(\frac{z_P - z}{\sin\theta}\right)\right]^2 = b^2\left|1 - \left|1 - \frac{z_P - z}{a\sin\theta}\right|^m\right|^{\frac{2}{n}} \quad \text{for } z_P \geq z \geq z_P - 2a\sin\theta, \text{ and}$$

$$\left[f\left(\frac{z_P - z}{\sin\theta}\right)\right]^2 = -b^2\left|1 - \left|1 - \frac{z_P - z}{a\sin\theta}\right|^m\right|^{\frac{2}{n}} \quad \text{for } z \geq z_P \text{ or } z \leq z_P - 2a\sin\theta.$$

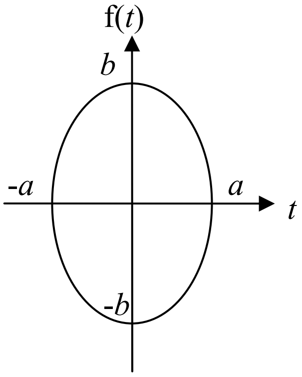
(See http://cda.morris.umn.edu/~kearnsk/Pages06-07/Math1011/Notes%20and%20Handouts/Power%20Laws-Solns.pdf.)

If $m = n = 2 \Rightarrow$ elliptical note.

If $m > 2$ and $n \geq 2 \Rightarrow$ an oval note (super ellipse) with "fatter" corners than an ellipse.

However graphing profiles (using excel speadsheets) show that for a smooth convex profile $F(z)$ at P, $n = 2$ which is another reason for $n = 2$. Probably in most cases $2 < m < 2.5$.

To graph the large note, rewrite the note equation with axis of symmetry on f(t) axis:



equ 3a:

$$f(t) = \pm b\left(1 - \left|\frac{t}{a}\right|^m\right)^{\frac{1}{n}}, \quad -a \leq t \leq a$$

Profile Equation using note equation, and n=2

$$x = \mathrm{F}(z) = r - \sqrt{\left(r - x_{\mathrm{P}} - \frac{z_{\mathrm{P}} - z}{\tan\theta}\right)^2 + \left[b\left(1 - \left|1 - \frac{z_{\mathrm{P}} - z}{a\sin\theta}\right|^m\right)^{\frac{1}{n}}\right]^2} \qquad \text{equ 1a}$$

If n=2, then $x = \mathrm{F}(z) = r - \sqrt{\left(r - x_{\mathrm{P}} - \frac{z_{\mathrm{P}} - z}{\tan\theta}\right)^2 + b^2\left(1 - \left|1 - \frac{z_{\mathrm{P}} - z}{a\sin\theta}\right|^m\right)}$ equ 1b

Derivative of Note Equation

$$\mathrm{f}\left(\frac{z_{\mathrm{P}} - z}{\sin\theta}\right) = \pm b\left(1 - \left(1 - \frac{z_{\mathrm{P}} - z}{a\sin\theta}\right)^m\right)^{\frac{1}{2}} \quad \text{if } z \ge z_{\mathrm{P}} - a\sin\theta \quad \text{equ3} \quad (\text{Only} + \text{needs to be considered})$$

$$\frac{d}{dz}\left[\mathrm{f}\left(\frac{z_{\mathrm{P}} - z}{\sin\theta}\right)\right] = \frac{b}{2}\left(1 - \left(1 - \frac{z_{\mathrm{P}} - z}{a\sin\theta}\right)^m\right)^{-\frac{1}{2}} \cdot -m\left(1 - \frac{z_{\mathrm{P}} - z}{a\sin\theta}\right)^{m-1} \cdot \frac{1}{a\sin\theta}$$

$$= \frac{-mb}{2a\sin\theta} \cdot \left(1 - \frac{z_{\mathrm{P}} - z}{a\sin\theta}\right)^{m-1} \cdot \left(1 - \left(1 - \frac{z_{\mathrm{P}} - z}{a\sin\theta}\right)^m\right)^{-\frac{1}{2}} \quad \text{when} \quad z > z_{\mathrm{P}} - a\sin\theta$$

$$= \frac{mb}{2a\sin\theta} \cdot \left(\frac{z_{\mathrm{P}} - z}{a\sin\theta} - 1\right)^{m-1} \cdot \left(1 - \left(\frac{z_{\mathrm{P}} - z}{a\sin\theta} - 1\right)^m\right)^{-\frac{1}{2}} \quad \text{when} \quad z < z_{\mathrm{P}} - a\sin\theta$$

$$= \frac{\mp mb^2}{2a\sin\theta} \frac{\left(\frac{z_{\mathrm{P}} - z}{a\sin\theta} - 1\right)^{m-1}}{\mathrm{f}\left(\frac{z_{\mathrm{P}} - z}{\sin\theta}\right)} \qquad \text{Use } -, \quad \text{when} \quad z > z_{\mathrm{P}} - a\sin\theta$$

Use +, when $z < z_{\mathrm{P}} - a\sin\theta$

Derivative of profile x=F(z)

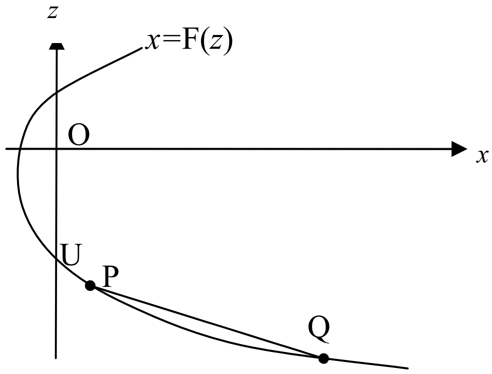
$$\mathrm{f}\left(\frac{z_{\mathrm{P}} - z}{\sin\theta}\right)\frac{d}{dz}\mathrm{f}\left(\frac{z_{\mathrm{P}} - z}{\sin\theta}\right) = \mp\frac{mb^2}{2a\sin\theta} \cdot \left|1 - \frac{z_{\mathrm{P}} - z}{a\sin\theta}\right|^{m-1}$$

$$\frac{dx}{dz} = \frac{\left(r - x_{\mathrm{P}} - \frac{z_{\mathrm{P}} - z}{\tan\theta}\right) + \tan\theta\left[\mathrm{f}\left(\frac{z_{\mathrm{P}} - z}{\sin\theta}\right)\right] \cdot \frac{d}{dz}\left[\mathrm{f}\left(\frac{z_{\mathrm{P}} - z}{\sin\theta}\right)\right]}{(x - r)\tan\theta}$$

$$= \frac{\left(r - x_{\mathrm{P}} - \frac{z_{\mathrm{P}} - z}{\tan\theta}\right) \mp \frac{mb^2}{2a\cos\theta} \cdot \left|1 - \frac{z_{\mathrm{P}} - z}{a\sin\theta}\right|^{m-1}}{(x - r)\tan\theta} \qquad \text{equ 2a}$$

Use −, when $z > z_{\mathrm{P}} - a\sin\theta$. Use +, when $z < z_{\mathrm{P}} - a\sin\theta$.

Find points P and Q, given U.



The profile shape x=F(z) passes through U, P, and Q.

At $\mathrm{U} = (x_{\mathrm{U}}, z_{\mathrm{U}})$, $\dfrac{dx}{dz} = \delta < 0$ and P≠U.

Most likely $x_{\mathrm{U}} = 0$ but doesn't have to be.

Using the derivative and profile equations already derived:

$$\delta = \frac{\left(r - x_{\mathrm{P}} - \dfrac{z_{\mathrm{P}} - z_{\mathrm{U}}}{\tan\theta}\right) - \dfrac{mb^2}{2a\cos\theta} \cdot \left|1 - \dfrac{z_{\mathrm{P}} - z_{\mathrm{U}}}{a\sin\theta}\right|^{m-1}}{(x_{\mathrm{U}} - r)\tan\theta} \qquad \text{equ 2c}$$

At U,

$$x = x_{\mathrm{U}} = \mathrm{F}(z_{\mathrm{U}}) = r - \sqrt{\left(r - x_{\mathrm{P}} - \frac{z_{\mathrm{P}} - z_{\mathrm{U}}}{\tan\theta}\right)^2 + b^2\left(1 - \left|1 - \frac{z_{\mathrm{P}} - z_{\mathrm{U}}}{a\sin\theta}\right|^m\right)} \qquad \text{equ 1c}$$

Given $x_{\mathrm{U}}, z_{\mathrm{U}}, \theta, \delta, m, a,$ and b, solve (2c) and (1c) to find z_{P}.

(2d) $$\left(r - x_{\mathrm{P}} - \frac{z_{\mathrm{P}} - z_{\mathrm{U}}}{\tan\theta}\right) = \delta(x_{\mathrm{U}} - r)\tan\theta + \frac{b^2 m}{2a\cos\theta} \cdot \left|1 - \frac{z_{\mathrm{P}} - z_{\mathrm{U}}}{a\sin\theta}\right|^{m-1}$$

(1d) $$0 = r - x_{\mathrm{U}} - \sqrt{\left[\delta(x_{\mathrm{U}} - r)\tan\theta + \frac{mb^2}{2a\cos\theta} \cdot \left|1 - \frac{z_{\mathrm{P}} - z_{\mathrm{U}}}{a\sin\theta}\right|^{m-1}\right]^2 + b^2\left(1 - \left|1 - \frac{z_{\mathrm{P}} - z_{\mathrm{U}}}{a\sin\theta}\right|^m\right)}$$

(1e) $$0 = (r - x_{\mathrm{U}})^2 - \left[\delta(x_{\mathrm{U}} - r)\tan\theta + \frac{mb^2}{2a\cos\theta} \cdot \left|1 - \frac{z_{\mathrm{P}} - z_{\mathrm{U}}}{a\sin\theta}\right|^{m-1}\right]^2 - b^2\left(1 - \left|1 - \frac{z_{\mathrm{P}} - z_{\mathrm{U}}}{a\sin\theta}\right|^m\right)$$

Solve equation (1e) for z_{P} (e.g. using techniques such as bisection method, false position method or secant method described in http://mathworld.wolfram.com/Root-FindingAlgorithm.html). When z_{P} is found, substitute into (2d) to find x_{P}. (If $z_{\mathrm{P}} > z_{\mathrm{U}}$, then alter *θ, m, a* or *b* until $z_{\mathrm{P}} < z_{\mathrm{U}}$.)

(2e) $$x_{\mathrm{P}} = r - \frac{z_{\mathrm{P}} - z_{\mathrm{U}}}{\tan\theta} - \delta(x_{\mathrm{U}} - r)\tan\theta - \frac{mb^2}{2a\cos\theta} \cdot \left|1 - \frac{z_{\mathrm{P}} - z_{\mathrm{U}}}{a\sin\theta}\right|^{m-1}$$

Or substitute z_{P} into (1c)

$$x_{\mathrm{P}} = r - \frac{z_{\mathrm{P}} - z_{\mathrm{U}}}{\tan\theta} - \sqrt{(r - x_{\mathrm{U}})^2 - b^2\left(1 - \left|1 - \frac{z_{\mathrm{P}} - z_{\mathrm{U}}}{a\sin\theta}\right|^m\right)} \qquad \text{equ 1f}$$

Can P=U, maximising θ? If P=U then (1b) is

$$0 = r - x_U - \delta(x_U - r)\tan\theta - \frac{mb^2}{2a\cos\theta}$$

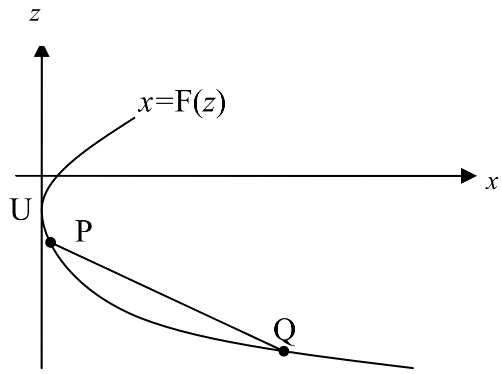
$$\gamma = \cos\theta + \delta\sin\theta \qquad \text{where} \qquad \gamma = \frac{mb^2}{2a(r - x_U)}$$

$$\gamma = \cos\theta + \delta\sqrt{1-\cos^2\theta}$$

$$0 = (1+\delta^2)\cos^2\theta - 2\gamma\cos\theta + \gamma^2 - \delta^2$$

$$\cos\theta_{max} = \frac{\gamma \pm \delta\sqrt{1+\delta^2-\gamma^2}}{1+\delta^2} \quad \text{where} \quad \delta = \frac{dx}{dz} \text{ at U} \quad \text{and} \quad \gamma = \frac{mb^2}{2a(r-x_U)}$$

If the bowl shape merges into the rim (which corresponds to the z-axis), then the z-axis is a tangent to the profile x=F(z).



At $U = (0, z_U)$, $\frac{dx}{dz} = \delta = 0$, and P≠U,

Substituting these values into (1e)

$$(1g) \qquad 0 = r^2 - \left(\frac{b^2 m}{2a\cos\theta}\right)^2 \cdot \left|1 - \frac{z_P - z_U}{a\sin\theta}\right|^{2(m-1)} - b^2\left(1 - \left|1 - \frac{z_P - z_U}{a\sin\theta}\right|^m\right)$$

Solve equation (1g) for z_P When z_P is found, substitute into (2c) to find x_P. (Note: If $z_P > z_U$, then alter θ, m, n, a or b until $z_P < z_U$.)

$$(2f) \qquad x_P = r - \frac{z_P - z_U}{\tan\theta} - \frac{b^2 m}{2a\cos\theta} \cdot \left|1 - \frac{z_P - z_U}{a\sin\theta}\right|^{m-1} \qquad \text{from 2c}$$

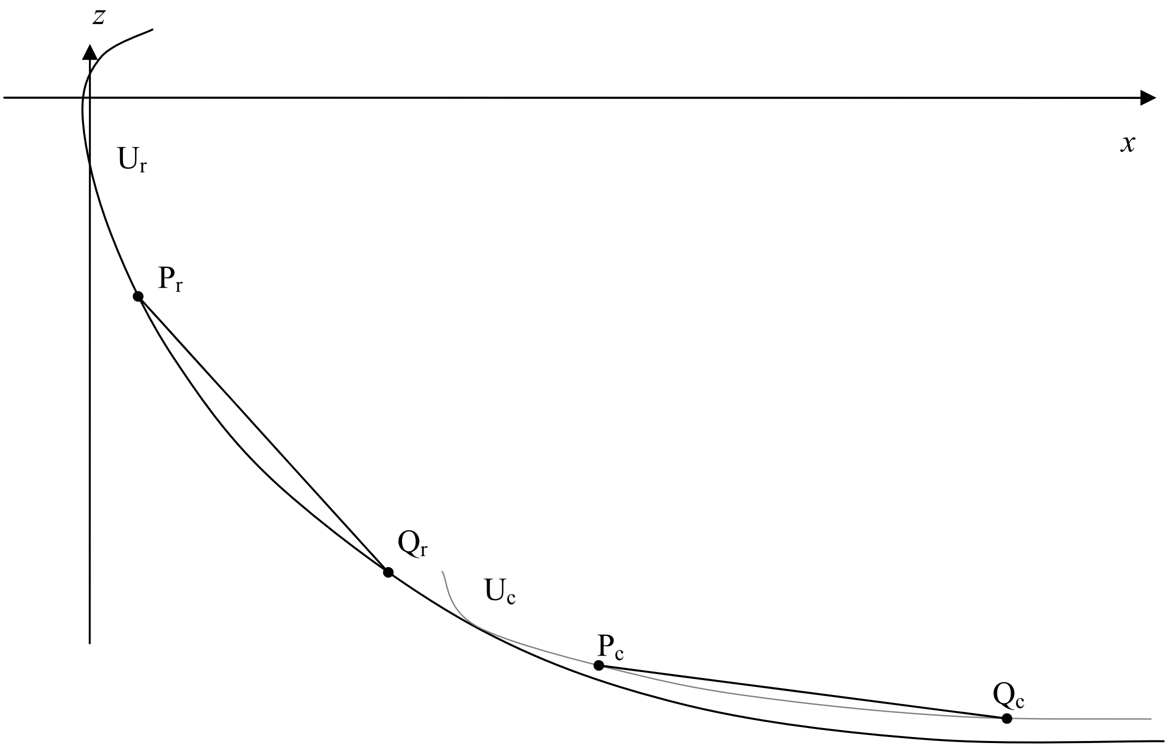
Or substitute z_P into (1f)

$$x_P = r - \frac{z_P - z_U}{\tan\theta} - \sqrt{r^2 - b^2\left(1 - \left|1 - \frac{z_P - z_U}{a\sin\theta}\right|^m\right)} \qquad \text{1h}$$

If P = U i.e. $x_P = x_U = 0$ and $z_P = z_U$ then (2f) is

$$\theta_{max} = \cos^{-1}\left(\frac{mb^2}{2ar}\right)$$

Centre Note P'Q'



To distinguish between rim and centre notes use subscripts r and c.
P_rQ_r and P_cQ_c lie on different curves. U_c is the point where the two curves blend together. Point U_c is the centre note's equivalent of the rim note's point U_r. At U_c the derivatives of the two curves are equal. (P_rQ_r and P_cQ_c have to be on different curves because the curve P_rQ_r extended to the centre (black line) will not produce a symmetrical note shape in the centre. So another curve has to be found for P_cQ_c (grey line).) Any edge point on the large centre note will have

$$x = r - \sqrt{\left(r - x_{\mathrm{Pc}} - \frac{z_{\mathrm{Pc}} - z}{\tan\theta c}\right)^2 + b_c^{\,2}\left(1 - \left|1 - \frac{z_{\mathrm{Pc}} - z}{a_c \sin\theta_c}\right|^{m_r}\right)} \qquad (1b)$$

Given $x_{\mathrm{U_C}} \geq x_{\mathrm{Qr}}$, calculate $z_{\mathrm{U_C}}$, or given $z_{\mathrm{U_C}} \leq z_{\mathrm{Qr}}$, calculate $x_{\mathrm{U_C}}$,

where $x_{\mathrm{U_C}} = r - \sqrt{\left(r - x_{\mathrm{Pr}} - \frac{z_{\mathrm{Pr}} - z_{\mathrm{U_C}}}{\tan\theta_r}\right)^2 + b_r^{\,2}\left(1 - \left|1 - \frac{z_{\mathrm{Pr}} - z_{\mathrm{U_C}}}{a_r \sin\theta_r}\right|^{m_r}\right)}$.

Calculate $\frac{d}{dz} x_{\mathrm{U_C}} = \delta_{\mathrm{U_C}} = \dfrac{r - x_{\mathrm{Pr}} - \frac{z_{\mathrm{Pr}} - z_{\mathrm{U_C}}}{\tan\theta_r} + \frac{m_r b_r^{\,2}}{2a_r\cos\theta_r}\cdot\left|1 - \frac{z_{\mathrm{Pr}} - z_{\mathrm{U_C}}}{a_r\sin\theta_r}\right|^{m_r - 1}}{\left(x_{\mathrm{U_C}} - r\right)\tan\theta_r}$.

(Use + (not -) in rim derivative equation because $z_{\mathrm{U_C}} < z_{\mathrm{Qr}} < z_{\mathrm{Pr}} - a_r \sin\theta_r$.)

Given centre note values a_c, b_c, m_c, n_c and θ_c, find point P_c.

At U_c, using centre note equations (which are the same as the rim notes, but use centre variables $a=a_c$, $b=b_c$, $m=m_c$, $n=n_c$ and $\theta=\theta_c$ instead of rim $a=a_r$, $b=b_r$, $m=m_r$, $n=n_r$ and $\theta=\theta_r$),

$$x_{U_C} = F_c(z_{U_C}) = r - \sqrt{\left(r - x_{Pc} - \frac{z_{Pc} - z_{U_C}}{\tan\theta_c}\right)^2 + b_c^2\left(1 - \left|1 - \frac{z_{Pc} - z_{U_C}}{a_C \sin\theta_c}\right|^{m_c}\right)} \qquad (1c)$$

Using the value of $\frac{d}{dz} x_{U_C} = \delta_{U_C}$ calculated above, the centre derivative at U is:

$$\delta_{U_C} = \frac{\left(r - x_{Pc} - \frac{z_{P_C} - z_{U_C}}{\tan\theta_C}\right) - \frac{m_C b_C^2}{2a_C \cos\theta_C} \cdot \left|1 - \frac{z_{Pc} - z_{U_C}}{a_C \sin\theta_C}\right|^{m_C - 1}}{\left(x_{U_C} - r\right)\tan\theta_C} \qquad (2c)$$

(1e) $$0 = \left(x_{U_C} - r\right)^2 - \left(\frac{m_C b_C^2}{2a_C \cos\theta_C}\left|1 - \frac{z_{Pc} - z_{U_C}}{a_C \sin\theta_C}\right|^{m_C - 1} + \delta_{U_C}\left(x_{U_C} - r\right)\tan\theta_C\right)^2 - b_c^2\left(1 - \left|1 - \frac{z_{Pc} - z_{U_C}}{a_C \sin\theta_c}\right|^{m_c}\right)$$

Solve (1e) for z_{Pc} and calculate x_{Pc} using

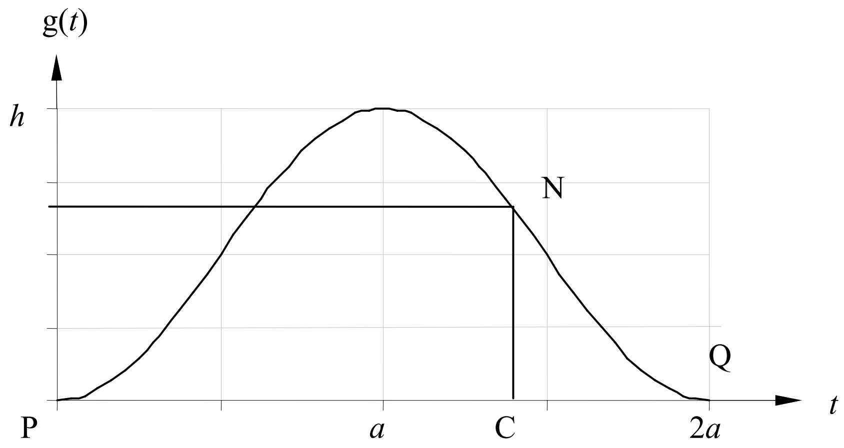
$$x_{Pc} = r - \frac{z_{P_C} - z_{U_C}}{\tan\theta_C} - \delta_{U_C}\left(x_{U_C} - r\right)\tan\theta_C - \frac{m_C b_C^2}{2a_C \cos\theta_C} \cdot \left|1 - \frac{z_{Pc} - z_{U_C}}{a_C \sin\theta_C}\right|^{m_C - 1} \qquad (2e)$$

or $$x_{Pc} = r - \frac{z_{Pc} - z_{U_C}}{\tan\theta_c} - \sqrt{\left(r - x_{U_C}\right)^2 - b_c^2\left(1 - \left|1 - \frac{z_{Pc} - z_{U_C}}{a_C \sin\theta_c}\right|^{m_c}\right)} \qquad (1f)$$

Maximum depth occurs at $x=r$ Substitute into (1b) and

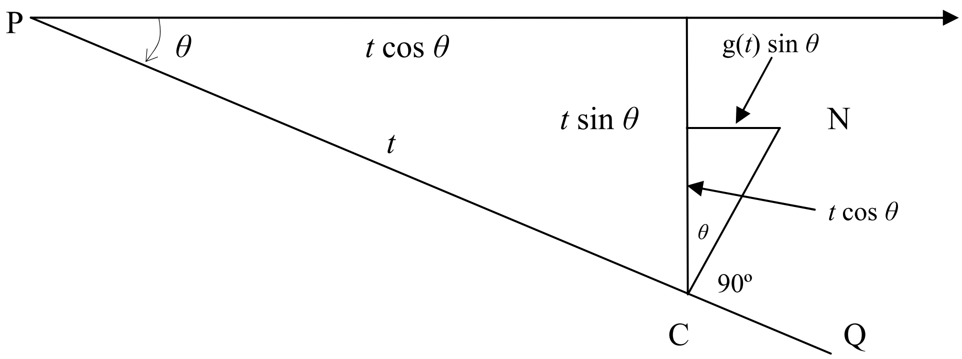
Solve for z_{max} $$0 = \left(r - x_{Pc} - \frac{z_{Pc} - z_{max}}{\tan\theta_C}\right)^2 + b^2\left(1 - \left|1 - \frac{z_{Pc} - z_{max}}{a_C \sin\theta_C}\right|^{m}\right)$$

Section through note surface



Vertical scale is greatly exaggerated compared to horizontal.

$$g(t) = \frac{h}{2}\left(1 - \cos\left(\frac{\pi t}{a}\right)\right) \qquad h \ll a$$

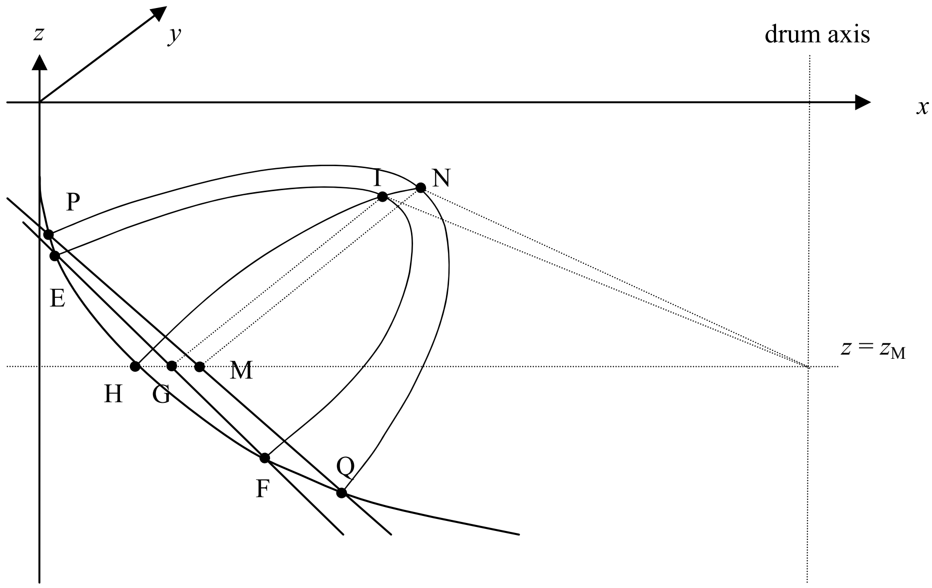


$$\begin{aligned} x_N &= x_C + g(t)\sin\theta \\ &= x_P + t\cos\theta + g(t)\sin\theta \\ &= x_P + t\cos\theta + \frac{h\sin\theta}{2}\left(1-\cos\left(\frac{\pi t}{a}\right)\right) \qquad (i) \end{aligned}$$

$$\begin{aligned} z_N &= z_C + g(t)\cos\theta \\ &= z_P - t\sin\theta + g(t)\cos\theta \\ &= z_P - t\sin\theta + \frac{h\cos\theta}{2}\left(1-\cos\left(\frac{\pi t}{a}\right)\right) \qquad (ii) \end{aligned}$$

Given x_N, $x_P \le x_N \le x_Q$, find t and hence calculate z_N.

Calculate the position of the smaller notes so that their edges lie on a plane, and are symmetrical on both axes.



(Note: This N is not the same N as in the previous diagram.)
M is the midpoint of PQ. G is the midpoint of EF. Points G, H, I, M and N lie on the plane $z=z_M$. PQ and MN are the axes of symmetry of the large note. EF and GI are the axes of symmetry of the small note. (I haven't mathematically proved that for the note EFI, GI lies on the same plane as MN. However I wrote a program that generated the note shapes produced when any plane EFI intersected with the profile curve PEFQ. These shapes were symmetrical only when $z_G = z_M$, and obviously EG=FG.) In other words, the horizontal axes of symmetry of all rim notes lie on the same horizontal (or depth or z) plane. The same applies for the centre notes. Incidentally, EF is not parallel to PQ.

Given the distance EF (i.e. the note width or length depending on its orientation), find the points E and F.

Points P, Q and M are known.

$$\mathrm{P} = (x_\mathrm{P}, z_\mathrm{P}) = (\mathrm{F}(z_\mathrm{P}), z_\mathrm{P}) \quad , \quad \mathrm{Q} = (x_\mathrm{Q}, z_\mathrm{Q}) = (\mathrm{F}(z_\mathrm{Q}), z_\mathrm{Q})$$

$$\mathrm{M} = (x_\mathrm{M}, z_\mathrm{M}) = \left(\frac{x_\mathrm{P} + x_\mathrm{Q}}{2}, \frac{z_\mathrm{P} + z_\mathrm{Q}}{2}\right) = \left(\frac{\mathrm{F}(z_\mathrm{P}) + \mathrm{F}(z_\mathrm{Q})}{2}, \frac{z_\mathrm{P} + z_\mathrm{Q}}{2}\right)$$

$$\mathrm{G} = (x_\mathrm{G}, z_\mathrm{G}) = \left(\frac{x_\mathrm{E} + x_\mathrm{F}}{2}, \frac{z_\mathrm{E} + z_\mathrm{F}}{2}\right) = \left(\frac{\mathrm{F}(z_\mathrm{E}) + \mathrm{F}(z_\mathrm{F})}{2}, \frac{z_\mathrm{P} + z_\mathrm{Q}}{2}\right)$$

$$z_\mathrm{G} = z_\mathrm{M}$$

$$\frac{z_\mathrm{E} + z_\mathrm{F}}{2} = \frac{z_\mathrm{P} + z_\mathrm{Q}}{2}$$

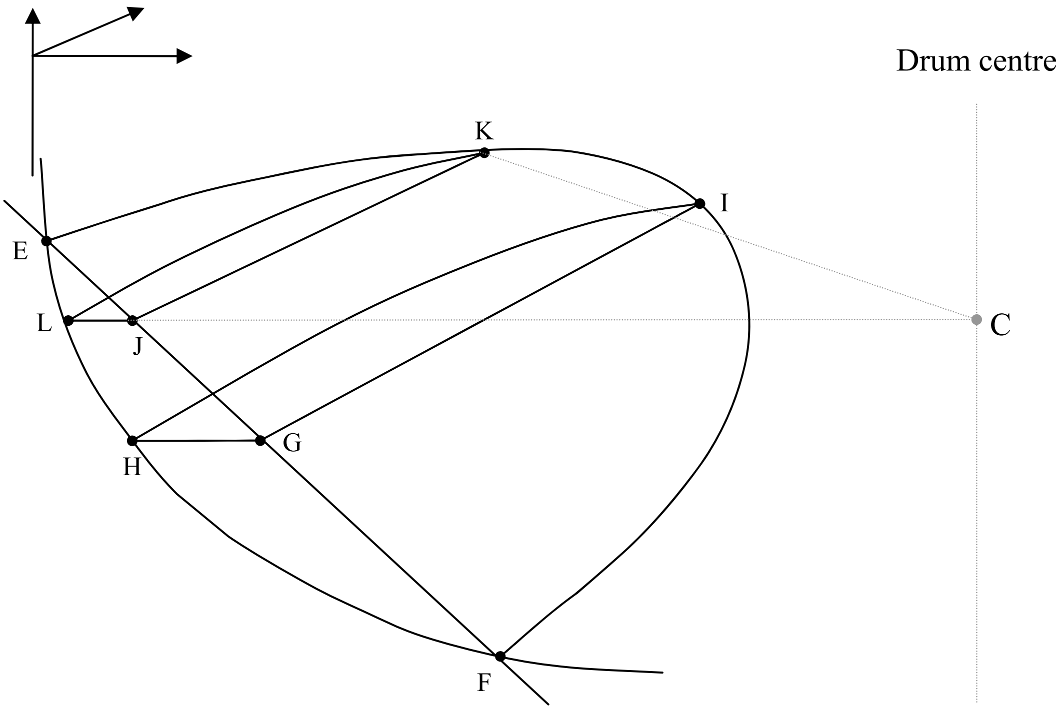
$$z_\mathrm{F} = z_\mathrm{P} + z_\mathrm{Q} - z_\mathrm{E}$$

$$\left(\frac{\mathrm{EF}}{2}\right)^2 = \mathrm{FG}^2$$

$$\begin{aligned}
\mathrm{FG}^2 &= (x_\mathrm{F} - x_\mathrm{G})^2 + (z_\mathrm{F} - z_\mathrm{G})^2 \\
&= \left(\mathrm{F}(z_\mathrm{F}) - \frac{\mathrm{F}(z_\mathrm{F}) + \mathrm{F}(z_\mathrm{E})}{2}\right)^2 + \left(z_\mathrm{P} + z_\mathrm{Q} - z_\mathrm{E} - \frac{z_\mathrm{P} + z_\mathrm{Q}}{2}\right)^2 \\
&= \left(\frac{\mathrm{F}(z_\mathrm{F}) - \mathrm{F}(z_\mathrm{E})}{2}\right)^2 + \left(\frac{z_\mathrm{P} + z_\mathrm{Q}}{2} - z_\mathrm{E}\right)^2 \\
&= \left(\frac{\mathrm{F}(z_\mathrm{P} + z_\mathrm{Q} - z_\mathrm{E}) - \mathrm{F}(z_\mathrm{E})}{2}\right)^2 + \left(\frac{z_\mathrm{P} + z_\mathrm{Q}}{2} - z_\mathrm{E}\right)^2 \\
\mathrm{EF}^2 &= \left(\mathrm{F}(z_\mathrm{P} + z_\mathrm{Q} - z_\mathrm{E}) - \mathrm{F}(z_\mathrm{E})\right)^2 + (z_\mathrm{P} + z_\mathrm{Q} - 2z_\mathrm{E})^2
\end{aligned}$$

Solve for z_E and calculate x_E, x_F, z_F, and x_G.

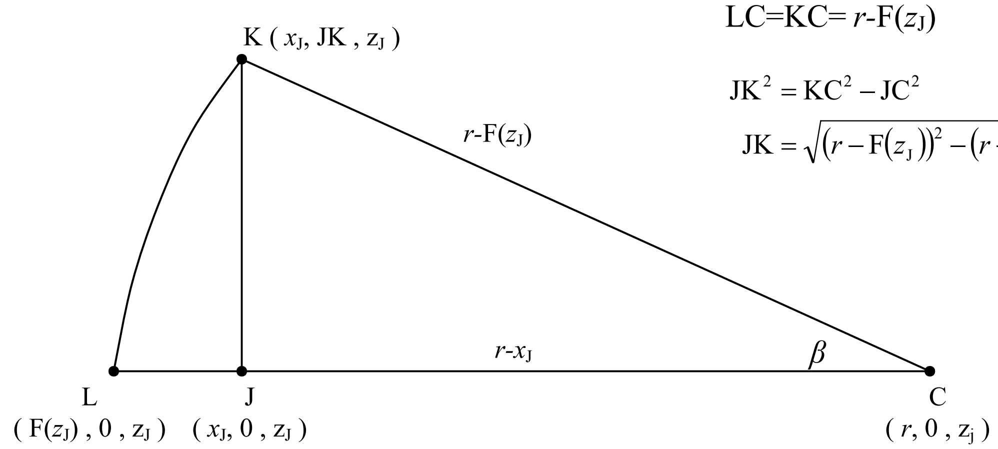
Determine shapes of small notes:



Equation of line EF is $z=mx+b$ or $x = \frac{z-b}{m}$. $m = \frac{z_\mathrm{E} - z_\mathrm{F}}{x_\mathrm{E} - x_\mathrm{F}}$ and $b = z_\mathrm{E} - m x_\mathrm{E}$.

The midpoint of EF is $\mathrm{G}(x_\mathrm{G}, 0, z_\mathrm{G}) = \left(\frac{x_\mathrm{E} + x_\mathrm{F}}{2}, 0, \frac{z_\mathrm{E} + z_\mathrm{F}}{2}\right)$.

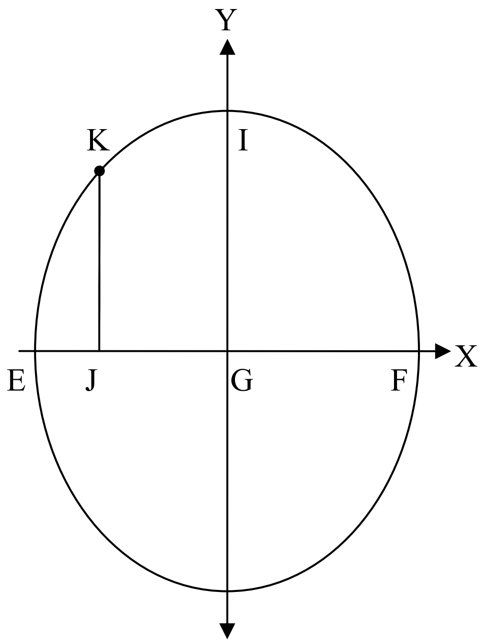
Given z_J, $\mathrm{J} = (x_\mathrm{J}, 0, z_\mathrm{J}) = \left(\frac{z_\mathrm{J} - b}{m}, 0, z_\mathrm{J}\right)$ and $\mathrm{JG} = \sqrt{(x_\mathrm{G} - x_\mathrm{J})^2 + (z_\mathrm{G} - z_\mathrm{J})^2}$.



LC=KC= r-F(z_J)

$$JK^2 = KC^2 - JC^2$$

$$JK = \sqrt{(r - F(z_J))^2 - (r - x_J)^2}$$



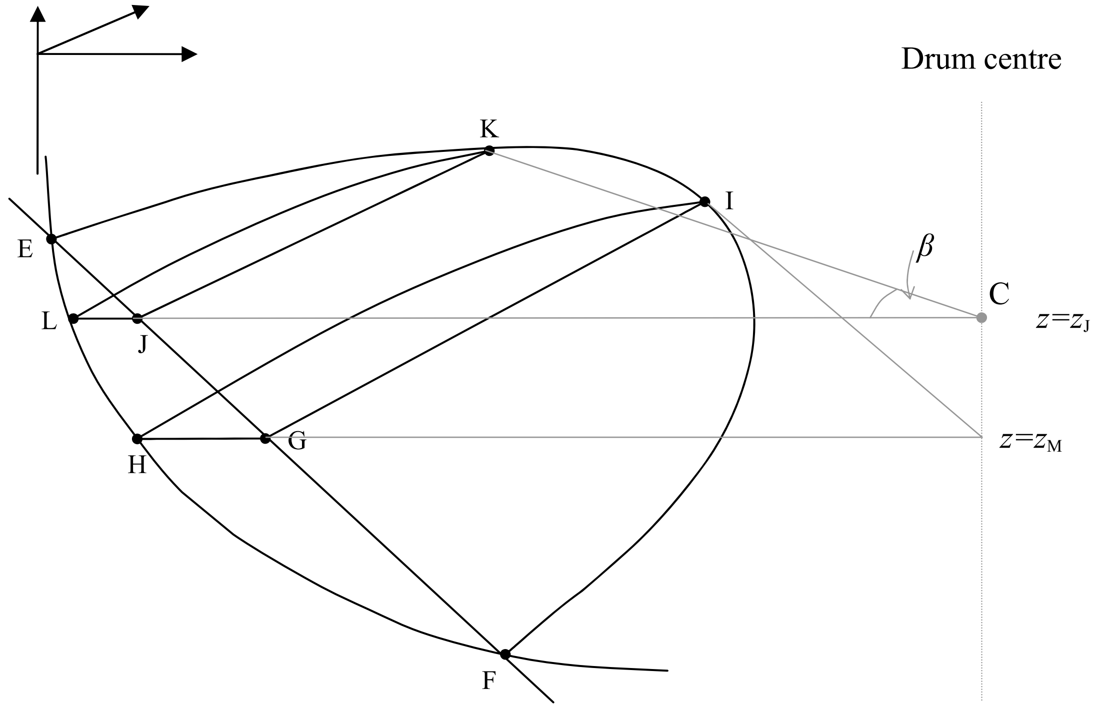
for $z_E \ge z_J \ge z_G$ $\quad JG = -\sqrt{(x_G - x_J)^2 + (z_G - z_J)^2}$

for $z_G \ge z_J \ge z_F$ $\quad JG = \sqrt{(x_G - x_J)^2 + (z_G - z_J)^2}$

for $z_E \ge z_J \ge z_F$ $\quad JK = \sqrt{(r - F(z_J))^2 - (r - x_J)^2}$

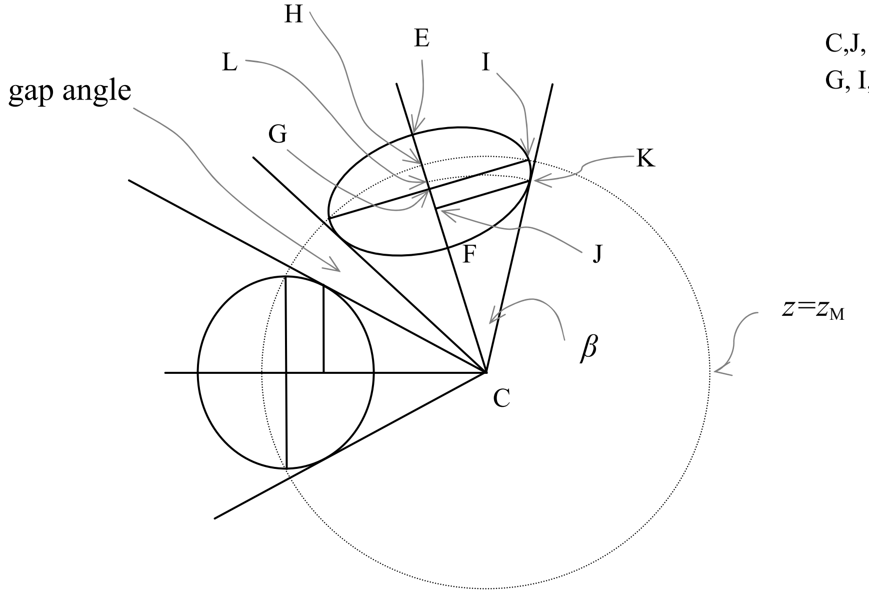
with x_J as above.

Note plane co-ordinates are K = (JG , JK) .



For any point K on the edge of the note, the angle at the centre C is $\angle \text{KCJ} = \beta$.

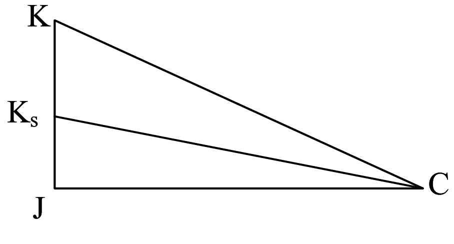
$\tan\beta = \dfrac{\text{JK}}{\text{JC}} = \dfrac{\text{JK}}{r - x_{\text{J}}}$ This angle is important for $z<z_M$. Finding the maximum angle for each note is used to find the average gap angle between notes.



C,J, K, and L are on z=z_J

G, I, and H are on z=z_M

Rewriting note edge as polar co-ordinates given $z=z_J$ and β is measured from the radial axis of symmetry: K= (KC, $\pm\beta$, z_J) where $z_F \leq z_J \leq z_E$



$\angle$JCK=β, $\angle$JCK$_S$=β_S

Given $\beta_S < \beta$, K_S (x, y, z)=(x_J , (r- x_J)tan β_S , z_J). K_s is inside the note. When $\beta_S = \beta$, K_S=K and is on the note edge or boundary.

$\text{K}_\text{S}\text{C} = \dfrac{\text{JC}}{\cos\beta_\text{S}} = \dfrac{r\text{-}x_\text{J}}{\cos\beta_\text{S}}$. So polar $\text{K}_\text{S} = \left(\text{K}_\text{S}\text{C}, \beta_\text{S}, z_\text{J}\right)$.